PROBLEMS AND SOLUTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

ALGEBRA.

396. Proposed by H. E. TREFETHEN, Colby College.

Show that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

GEOMETRY.

424. Proposed by H. E. TREFETHEN, Colby College.

In a given triangle ABC, determine by geometric demonstration the point O so that the sum of the distances, $AO + BO + CO$, shall be a minimum.

425. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the ratio of the areas, A_1 and A_2, of the parabolas formed by projectiles whose ranges are the same and whose angles of projection are complements of each other.

CALCULUS.

345. Proposed by C. N. SCHMALL, New York City.

"Of all the quadrilaterals which can be formed from four given sides, that which is inscriptible in a circle has the greatest area."

[From Goursat-Hedrick, *Math. Anal.*, p. 133, ex. 5.]

Proposer's Remark.—Prove this, and furthermore show that when only three sides are known, the length of the fourth side of the maximum quadrilateral is the root of a cubic equation.

346. Proposed by C. N. SCHMALL, New York City.

Given the height of an inclined plane, to find its length so that a given force, acting on a given mass in a direction parallel to the plane, may draw it up in the shortest time.

Note.—This question has a practical application in the case of a truck, of known height, upon which a mass is to be loaded by means of skids.

MECHANICS.

280. Proposed by C. N. SCHMALL, New York City.

Given the distance d between two smooth hooks in the same horizontal line. Show that the shortest string which can form a catenary, with these hooks for points of support, is de, where e is the base of the Naperian system of logarithms.

281. Proposed by C. N. SCHMALL, New York City.

ABC is a triangle inscribed in a circle, center O, and L, M, N, are the centers of gravity of the sectors AOB, BOC, COA. Show that

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3\pi.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

197. Proposed by E. T. BELL, Seattle, Wash.

Show that in the expansion of $\dfrac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1$, where p is a prime, the coefficients of the various powers of z are divisible by p. [Eisenstein, *Crelle, t. 27, p. 282.*]

198. Proposed by ARTEMAS MARTIN, Washington, D. C.

Prove that every even number is the sum of two prime numbers.

Note.—This problem has long been known and no proof has ever been given. [EDITORS.]

CORRECTION.—Number **192,** incorrectly given as **188** in the June issue, page 196, should have the zeros on the right of equations (1), (2), (3) each replaced by $\square$, the symbol for a "square number."

SOLUTIONS OF PROBLEMS.

ALGEBRA.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots, u_1, u_2, u_3, \cdots$, show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n \doteq \infty} = \frac{1}{2}.$$

SOLUTION BY H. A. LEVY, Houghton, Mich.

Consider the sequence $\frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \cdots, \frac{q-1}{q}$.

(1) Suppose q to be an odd prime. Then the quantities $\frac{n}{q}$ ($n = 1, 2, 3, \cdots, q-1$) are irreducible fractions, and $q-1$ must be an even number, say $2k$. Then

$$\sum_{n=1}^{q-1} \frac{n}{q} = \frac{1+2+3+\cdots+(q-1)}{q} = \frac{q-1}{2} = k.$$

Hence, in such a sequence, there are $2k$ terms, and the sum of the terms is k.

(2) Suppose that q is an even number of the form $2^r p$, where p is any odd prime except one. All the fractions with even numerators are then reducible, and the sequence is

$$\frac{1}{2^r p}, \frac{3}{2^r p}, \frac{5}{2^r p}, \cdots, \frac{2^r p - 1}{2^r p}.$$

In this sequence, at least one term is reducible, viz., $\frac{p}{2^r p}$. If there are more, they are the terms $\frac{p}{2^r p}, \frac{3p}{2^r p}, \frac{5p}{2^r p}, \cdots$ up to the greatest odd multiple of p less than $2^r p$. This odd multiple must therefore be of the form $4k+1$. Hence an odd number of terms are reducible. Furthermore, the numerator of the last term of the sequence, $2^r p - 1$, is of the form $4k+1$, so that there is an odd number of terms in the sequence. Rejecting the reducible terms, therefore, leaves an even number of irreducible terms.

If $r = 1$, the number of terms is $p - 1$, an even number. The sum of the terms is then